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Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves

By N. SUGIMOTO
Department of Mechanical Engineering, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan

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This paper deals with initial-value problems for the Burgers equation with the inclusion of a hereditary integral known as the fractional derivative of order $\frac{1}{2}$. Emphasis is placed on the difference between the local and global dissipation due to the second-order and the half-order derivatives, respectively. Exploiting the smallness of the coefficient of the second-order derivative, an asymptotic analysis is first developed. When a discontinuity appears, the matched-asymptotic expansion method is employed to derive a uniformly valid solution. If the coefficient of the half-order derivative is also small, as is usually the case, the evolution comprises three stages, namely a lossless near field, an intermediate Burgers region, and a hereditary far field. In view of these results, the equation is then solved numerically, under various initial conditions, by finite-difference and spectral methods. It is revealed that the effect of the fractional derivative accumulates slowly to give rise to a significant dissipation and distortion of the waveform globally, which is to be contrasted with the effect of the second-order derivative, significant only locally, in a thin 'shock layer'.

1. Introduction

This paper deals with initial-value problems for the Burgers equation with the inclusion of a hereditary integral known as the fractional derivative of order $\frac{1}{2}$:

$$\frac{\partial f}{\partial X} - \alpha f \frac{\partial f}{\partial \theta} = \beta \frac{\partial^2 f}{\partial \theta^2} - \delta \frac{\partial^{\frac{1}{2}} f}{\partial \theta^{\frac{1}{2}}}, \tag{1.1}$$

subject to the initial condition

$$f(\theta, X = 0) = F(\theta), \tag{1.2}$$

where the half-order derivative is defined by

$$\frac{\partial^{\frac{1}{2}} f}{\partial \theta^{\frac{1}{2}}} = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\theta} \frac{1}{(\theta - \theta')^{\frac{1}{2}}} \frac{\partial f(\theta', X)}{\partial \theta'} \mathrm{d}\theta'. \tag{1.3}$$

Equation (1.1) describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the cumulative (memory) effect of the wall friction through the boundary layer (Chester 1964; Keller 1981, Sugimoto 1989). (Generally speaking, a boundary layer will give rise to memory effects in the form of this fractional derivative. The same form can be found in other systems such as shallow-water waves (Kakutani & Matsuuchi 1975) and waves in bubbly liquids (Miksis & Ting 1990).) The remaining terms have the same physical meanings as in the usual Burgers equation except for the definition of the independent variables. Here X and θ denote, respectively, the spatial coordinate and the retarded time measured in a frame moving with the sound speed, so that (1.1) describes the spatial evolution of a fluid velocity f, appropriately normalized. Although, therefore, the condition (1.2) prescribes physically a boundary value, it is simply referred to here as the 'initial' condition.

In (1.1), α denotes the well-known nonlinear coefficient $\frac{1}{2}(\gamma+1)$, γ being the ratio of specific heats. The dissipation constant β due to the diffusivity of sound is far smaller than the other constant δ due to the wall friction ($0 < \beta \ll \delta \lesssim 1$). In fact, β/δ is of order $R^{-\frac{1}{2}}D/L$, where R ($\gg 1$) denotes the acoustic Reynolds number and D and L denote, respectively, the diameter of the pipe and a characteristic wavelength (for the details, see Sugimoto 1989). Since L is usually greater than D, the second-order derivative may be ignored in the first approximation. The steady-progressive-wave solution and the solutions to this equation for some typical initial-value problems show that the propagation of a discontinuity plays an essential role (Sugimoto 1989, 1990). Although this local property is to be compared with that of a hyperbolic wave equation, the fractional derivative exhibits a slow relaxation characteristic of the hereditary integral, giving rise to a significant dissipation and distortion of waveform globally.

When the discontinuity appears, however, its local characteristic wavelength vanishes, so that $\beta f_{\theta\theta}$ no longer remains small. Just as in the Burgers equation, the second-order derivative then comes into play to replace the discontinuity by a thin 'shock layer'. In the 'outer region' surrounding this layer, however, its effect remains secondary, behind that of the hereditary one. Thus the two dissipations play different roles in the evolution. This difference is also well illustrated from the standpoint of the asymptotic analysis in terms of the small dissipation parameters.

In what follows, we first develop the asymptotic analysis in terms of β. When the discontinuity appears, a matched-asymptotic expansion method is employed to derive a uniformly valid expansion. The asymptotic analysis enables us to preview the evolution behaviour before embarking on solving the equation numerically. For numerical calculations, two different methods are used, depending on the type of initial condition. The finite-difference method is applied to localized conditions for which the hereditary integral converges (see, for example, Mitchell & Griffiths 1980). To periodic conditions, however, for which the integral should be defined in the sense of generalized functions, the spectral (Galerkin) method is applied (Gazdag 1973; Basdevant *et al.* 1988; Sachdev 1987). It is shown that the hereditary effect significantly modifies the solutions of the Burgers equation, especially in the far field.

2. Asymptotic analysis

In this section, we consider the evolution of f from the standpoint of asymptotic analysis in terms of the small parameter β. In this context, the other parameter δ need not be small. The nonlinear coefficient α is set equal to unity in the following by rescaling f.

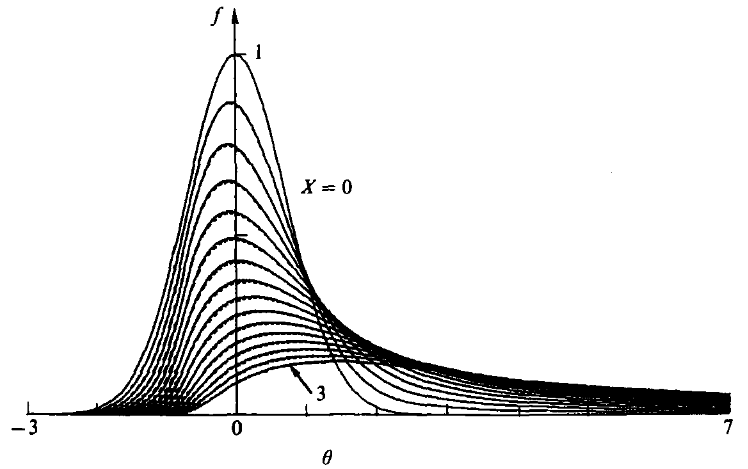

FIGURE 1. Evolution from the positive Gaussian-shaped pulse: $F(\theta) = \exp(-\theta^2)$; the solid lines represent the evolution of f_0 by (2.2) with $\delta = 1$, i.e. f by (1.1) with $\beta = 0$ and $\delta = 1$ up to $X = 3$ by steps in X of 0.2, while the broken lines represent that by (1.1) with $\beta = 0.01$ and $\delta = 1$.

2.1. *Regular expansion*

It is natural, first, to seek a solution to (1.1) in the form of a naive expansion in powers of β:

$$f = f_0 + \beta f_1 + O(\beta^2). \tag{2.1}$$

The lowest-order problem is to solve

$$\frac{\partial f_0}{\partial X} - f_0 \frac{\partial f_0}{\partial \theta} = -\delta \frac{\partial^{\frac{1}{2}} f_0}{\partial \theta^{\frac{1}{2}}}. \tag{2.2}$$

The initial condition is taken as

$$f_0(\theta, X = 0) = F(\theta). \tag{2.3}$$

If the solution f_0 is available, f_1 is then determined by

$$\frac{\partial f_1}{\partial X} - \frac{\partial}{\partial \theta}(f_0 f_1) = \frac{\partial^2 f_0}{\partial \theta^2} - \delta \frac{\partial^{\frac{1}{2}} f_1}{\partial \theta^{\frac{1}{2}}}, \tag{2.4}$$

under the initial condition $f_1(\theta, X = 0) = 0$. Proceeding in this way, equations governing higher-order terms in (2.1) can be derived systematically.

To solve (2.2) by analogy with the hyperbolic equation ($\delta = 0$), it can be expressed in the 'characteristic form':

$$\frac{\mathrm{d} f_0}{\mathrm{d} X} = -\delta \frac{\partial^{\frac{1}{2}} f_0}{\partial \theta^{\frac{1}{2}}} \tag{2.5}$$

along the 'characteristics' defined by

$$\frac{\mathrm{d}\theta}{\mathrm{d}X} = -f_0. \tag{2.6}$$

But this representation is formal in the sense that (2.5) is not reducible to an ordinary differential equation to be satisfied along the characteristics. Nevertheless this form has merit for integrating (2.5) numerically along (2.6).

Sugimoto (1990) demonstrated numerically four typical evolutions of (2.2) from the positive and negative steps $F(\theta) = \pm h(\theta)$, $h(\theta)$ being the unit step function, and the positive and negative Gaussian-shaped pulses $F(\theta) = \pm\exp(-\theta^2)$. For a large value of δ, comparable with unity or greater, the dissipative effect involved in the half-order derivative suppresses the nonlinear steepening, so that a smooth initial condition evolves smoothly without forming a discontinuity. For the Gaussian-shaped pulses, the evolutions for $\delta = 1$ are typical examples. Figures 1 and 2 show their evolution, in which the solid lines represent the solutions to (2.2), while the broken lines represent, for reference, those to the full equation (1.1) with $\beta = 10^{-2}$. It is seen that (1.1) is well approximated by (2.2). In figure 2, in particular, the two lines almost coincide. While f_0 remains smooth, there would be no difficulty in solving for the higher-order terms. As δ becomes small, however, it is usually the case that the nonlinear steepening cannot be balanced by the fractional derivative, so that f_0 becomes multi-valued in the course of evolution. When this multi-valuedness emerges, a discontinuity must be fitted into the solution to make f_0 single-valued. In the case, incidentally, when the initial condition itself contains a discontinuity, it may of course be propagated from the outset.

When the discontinuity appears, it has been mentioned that the second-order term $\beta f_{\theta\theta}$ no longer remains of order β. The regular expansion then breaks down around the discontinuity. But it still remains valid in a region away from it. In order to render the expansion uniformly valid, the discontinuity should be replaced by a thin but smooth shock layer. With the whole region split into this shock layer and the remaining outer region, the matched-asymptotic expansion method is employed to connect both regions. This procedure will be explained in the next section.

2.2. *Matched-asymptotic expansion*

2.2.1. *Outer expansion*

Upon assuming that a discontinuity already exists in the outer solution of (2.2), we first examine relations for the discontinuity to satisfy. Let the discontinuity be located at $\theta = \tau(X)$ and let η be defined by $\theta - \tau(X)$ so as to take the origin of η at the discontinuity. Because τ may involve β, it is assumed to be expanded in terms of β:

$$\tau = \tau_0(X) + \beta\tau_1(X) + O(\beta^2). \tag{2.7}$$

Rewriting (1.1) in terms of η and X and using the expansions (2.1) and (2.7), the lowest-order problem in powers of β becomes

$$\frac{\partial f_0}{\partial X} - \dot{\tau}_0\frac{\partial f_0}{\partial \eta} - \frac{\partial}{\partial \eta}\left(\frac{f_0^2}{2}\right) = -\delta\frac{\partial^{\frac{1}{2}} f_0}{\partial \eta^{\frac{1}{2}}}, \tag{2.8}$$

where $\dot{\tau}_0 = \mathrm{d}\tau_0/\mathrm{d}X$ and the dot implies differentiation with respect to X hereafter. Here f_0 stands for $f_0(\eta, X)$, since the argument θ in (2.1) has been changed to η. On the left side of the discontinuity, i.e. $\eta < 0$, let a continuous solution $f_0 = B(\eta, X)$ prevail. Then f_0 is assumed to be expressed, with a local behaviour around and including the discontinuity, as

$$f_0 = B(\eta, X) + [V_0 + V_1|\eta|^{\frac{1}{2}} + V_2|\eta| + \ldots + V_n|\eta|^{n/2} + \ldots]\,h(\eta), \tag{2.9}$$

where $h(\eta)$ is the unit step function and $-\infty < \eta \ll 1$; $V_n = V_n(X)$ $(n = 0, 1, 2, \ldots)$ and

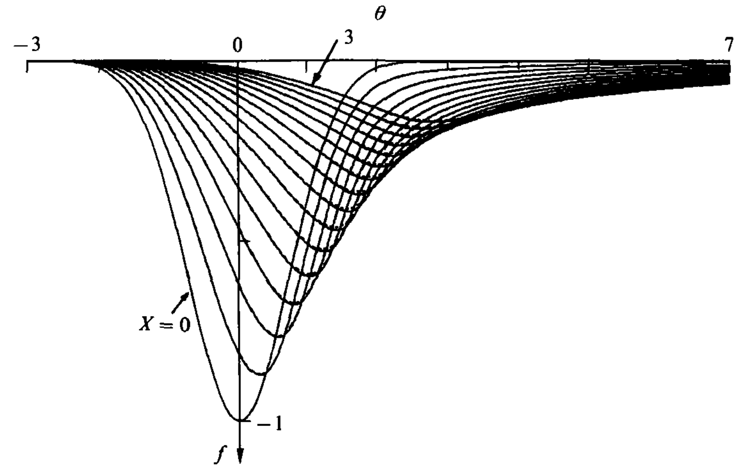

FIGURE 2. Evolution from the negative Gaussian-shaped pulse: $F(\theta) = -\exp(-\theta^2)$; the solid lines represent the evolution of f_0 by (2.2) with $\delta = 1$, i.e. f by (1.1) with $\beta = 0$ and $\delta = 1$ up to $X = 3$ by steps in X of 0.2, while the broken lines represent that by (1.1) with $\beta = 0.01$ and $\delta = 1$.

$V_0 (\neq 0)$ gives the strength of the discontinuity. Here the half powers of $|\eta|$ are assumed to be present in view of the formula of fractional derivatives:

$$\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}\eta^{\frac{1}{2}}}\frac{|\eta|^p h(\eta)}{\Gamma(p+1)} = \frac{|\eta|^{p-\frac{1}{2}} h(\eta)}{\Gamma(p+\frac{1}{2})}, \tag{2.10}$$

where $p = \frac{1}{2}n$ $(n = 0, 1, 2, \ldots)$ and $\Gamma(p)$ denotes the Gamma function. Suppose that $B(\eta, X)$ can be expanded appropriately around $\eta = 0$ so that it is continued beyond $\eta = 0$:

$$B(\eta, X) = B_0 + B_2\,\eta + \ldots + B_{2n}\,\eta^n + \ldots, \tag{2.11}$$

where $B_{2n} = B_{2n}(X)$ $(n = 0, 1, 2, \ldots)$ and B_0 gives the limiting value of f approaching the discontinuity from the left.

Let us now introduce (2.9) into (2.8). Of course, B satisfies (2.8) in $\eta < 0$. For the quadratic term in f_0, (2.11) is used and $h(\eta)^2$ is set equal to $h(\eta)$. Also, use is made of the relations

$$\eta^m |\eta|^{n/2} h(\eta) = |\eta|^{m+n/2} h(\eta) \quad (m, n\text{: non-negative integers}),$$

$$(\mathrm{d}/\mathrm{d}\eta)\, h(\eta) = \delta(\eta), \quad (\mathrm{d}/\mathrm{d}\eta)\, |\eta|^{n/2} h(\eta) = (\tfrac{1}{2}n)\, |\eta|^{n/2-1} h(\eta) \quad (\eta \neq 0),$$

where $\delta(\eta)$ denotes the delta function. The fractional derivative of B is assumed to be expanded around $\eta = 0$ as

$$\frac{\partial^{\frac{1}{2}} B}{\partial \eta^{\frac{1}{2}}} = \left.\frac{\partial^{\frac{1}{2}} B}{\partial \eta^{\frac{1}{2}}}\right|_{\eta=0} + \left.\frac{\partial^{\frac{3}{2}} B}{\partial \eta^{\frac{3}{2}}}\right|_{\eta=0} \eta + \ldots + \left.\frac{\partial^{n+\frac{1}{2}} B}{\partial \eta^{n+\frac{1}{2}}}\right|_{\eta=0} \frac{\eta^n}{n!} + \ldots, \tag{2.12}$$

where the fractional derivatives of higher order, $\partial^{n+\frac{1}{2}} B / \partial \eta^{n+\frac{1}{2}}$ $(n = 1, 2, 3, \ldots)$, are defined by differentiating (1.3) with respect to η (Sugimoto 1989). For the discontinuous parts in (2.9), the formula (2.10) is used. After such preparations, we substitute (2.9) into (2.8) to have immediately from the coefficients of $\delta(\eta)$

$$\frac{\mathrm{d}\tau_0}{\mathrm{d}X} = -(B_0 + \tfrac{1}{2}V_0). \tag{2.13}$$

Using (2.12) and (2.13), the coefficients B_{2n} in (2.11) are determined, by expanding (2.8) for $f_0 = B$ around $\eta = 0$, as follows:

$$\left.\begin{aligned} B_2 &= -\frac{2}{V_0}\left(\dot{B}_0 + \delta \frac{\partial^{\frac{1}{2}} B}{\partial \eta^{\frac{1}{2}}}\bigg|_{\eta=0}\right), \\ B_{2n} &= -\frac{2}{nV_0}\bigg[\dot{B}_{2n-2} - \tfrac{1}{2}n(B_2 B_{2n-2} + B_4 B_{2n-4} + \dots \\ &\qquad + B_{2n-2}B_2) + \frac{\delta}{(n-1)!}\frac{\partial^{n-\frac{1}{2}} B}{\partial \eta^{n-\frac{1}{2}}}\bigg|_{\eta=0}\bigg] \quad (n = 2, 3, 4, \dots). \end{aligned}\right\} \tag{2.14}$$

On the other hand, the V_n $(n = 1, 2, 3, \dots)$ are determined from the coefficients of $|\eta|^{n/2-1}h(\eta)$ $(n = 1, 2, 3, \dots)$ as follows:

$$\left.\begin{aligned} V_1 &= \frac{4\delta}{\pi^{\frac{1}{2}}}, \quad V_2 = \frac{2}{V_0}\left[\dot{V}_0 + 2\delta^2\left(1 - \frac{4}{\pi}\right)\right] - 2B_2, \\ V_n &= \frac{4}{nV_0}\bigg[\dot{V}_{n-2} - \tfrac{1}{4}n(V_1 V_{n-1} + V_2 V_{n-2} + \dots + V_{n-1} V_1) \\ &\quad + \delta \frac{\Gamma[\frac{1}{2}(n+1)]}{\Gamma(\frac{1}{2}n)} V_{n-1} - \tfrac{1}{2}n(B_2 V_{n-2} + B_4 V_{n-4} + \dots + B_n V_0)\bigg] \quad (n = 3, 4, 5, \dots), \end{aligned}\right\} \tag{2.15}$$

where the B_n vanish for odd n. Thus we have, albeit formally, determined the relations for the discontinuity to satisfy.

Relation (2.13) determines the propagation velocity of the discontinuity, which is simply the one derived from the hyperbolic equation ($\delta = 0$). In line with the general property that the highest derivative involved in an equation usually determines the qualitative behaviour of its solution, the order $\frac{1}{2}$ of the fractional derivative is indeed lower than the highest one in (2.2), i.e. the first order. Effects of the fractional derivative first appear in V_1. Interestingly, however, V_1 is an absolute constant independent of V_0. Because of the term $V_1|\eta|^{\frac{1}{2}}$ in (2.9), the discontinuity appears to be rounded rightward, but sharp-edged leftward. In the numerical calculations by Sugimoto (1990), (2.9) and (2.11) are taken up to $O(\eta)$, and (2.2) is solved rather than (2.8) because τ_0, B_0 and V_0 must be sought as part of the solution.

As regards the expansion (2.9), we note the non-uniformity as V_0 vanishes. For (2.9) to be valid, it is required that $V_{n+1}|\eta|^{\frac{1}{2}}/V_n \sim o(1)$ $(n = 0, 1, 2, \dots)$ as η tends to zero. If V_0 tends to vanish, the region of validity for the η variable becomes narrower, i.e. $0 < \eta \ll V_0^2$, provided the derivatives of V_n with respect to X remain finite there. This non-uniformity implies that some other type of expansion than (2.9) must be prepared. This will be shown in §2.3. In this connection, we remark that (2.9) also exhibits another type of non-uniformity when the derivatives of V_0 with respect to X diverge. We explain this situation by the specific example of the evolution from the positive step $F(\theta) = h(\theta)$. Since V_1 is an absolute constant, then obviously, (2.9) with $B = 0$ cannot represent the initial step at $X = 0$, even if V_0 were chosen to be unity. The asymptotic analysis in the Appendix shows that the magnitude V of the discontinuity at the wavefront is given by $V = 1 - \delta(8X/\pi)^{\frac{1}{2}} + O(X)$ as $X \to 0$, so that $\mathrm{d}V/\mathrm{d}X$ diverges in this limit. (Here V should be distinguished from V_0 in (2.9). As far as the expansion (2.9) is legitimate, V is equal to V_0, of course.) If V_0 were taken to be V, V_2 in (2.15) would make (2.9) non-uniform. In fact, (A 1) with (A 4) and (A 5) in the Appendix suggests an expansion different from (2.9).

Let us now return to seek the higher-order terms in the expansion (2.1). Equation (2.4) then becomes

$$\frac{\partial f_1}{\partial X}-\dot{\tau}_0\frac{\partial f_1}{\partial \eta}-\dot{\tau}_1\frac{\partial f_0}{\partial \eta}-\frac{\partial}{\partial \eta}(f_0 f_1)=\frac{\partial^2 f_0}{\partial \eta^2}-\delta\frac{\partial^{\frac{1}{2}} f_1}{\partial \eta^{\frac{1}{2}}}. \tag{2.16}$$

Since f_0 is subjected to a discontinuity, so also will be the case with f_1. In view of (2.9), f_1 is assumed to be similarly expressed as the sum of a continuous solution $b(\eta, X)$ to (2.16) in $\eta < 0$ and of discontinuous parts:

$$f_1 = b(\eta,X)+[v_{-2}|\eta|^{-1}+v_{-1}|\eta|^{-\frac{1}{2}}+v_0+\dots]\,h(\eta), \tag{2.17}$$

where $v_n = v_n(X)\,(n=-2,-1,0)$. Here we assume that $b(\eta,X)$ can be expanded around $\eta = 0$ so that it can be continued beyond it as

$$b = b_0+b_2\eta+\dots, \tag{2.18}$$

where $b_{2n} = b_{2n}(X)\,(n = 0,1)$. In passing, substitution of (2.18) into (2.16) yields

$$b_2 = -\frac{2}{V_0}\left[\dot{b}_0-(b_0+\dot{\tau}_1)B_2+\delta\left.\frac{\partial^{\frac{1}{2}} b}{\partial \eta^{\frac{1}{2}}}\right|_{\eta=0}-2B_4\right]. \tag{2.19}$$

In (2.17), the singular terms $v_{-2}|\eta|^{-1}h(\eta)$ and $v_{-1}|\eta|^{-\frac{1}{2}}h(\eta)$ must be incorporated to balance the singularity due to the second-order derivative of f_0 in (2.16). Here we recall the following important relation which provides an alternative definition of the delta function (Gel'fand & Shilov 1964, p. 57):

$$\delta(\eta) = \frac{|\eta|^{-1}}{\Gamma(0)}h(\eta). \tag{2.20}$$

By virtue of this, we have, on substituting (2.17) into (2.16), from the coefficients of $\mathrm{d}\delta(\eta)/\mathrm{d}\eta$, $|\eta|^{-\frac{3}{2}}h(\eta)$ and $\delta(\eta)$, respectively:

$$v_{-2} = -\frac{2}{\Gamma(0)}, \quad v_{-1} = -\frac{8\delta}{\pi^{\frac{1}{2}}V_0}, \tag{2.21\,a, b}$$

and

$$\dot{\tau}_1 = -b_0-\tfrac{1}{2}v_0-\frac{2}{V_0^2}\left[2\dot{B}_0+\dot{V}_0+2\delta\left.\frac{\partial^{\frac{1}{2}} B}{\partial \eta^{\frac{1}{2}}}\right|_{\eta=0}+6\delta^2\left(1-\frac{4}{\pi}\right)\right]. \tag{2.21\,c}$$

Because $\Gamma(0)$ is infinite, v_{-2} is substantially zero. But when combined with $|\eta|^{-1}$ in (2.17), it comes into effect to yield the delta function. The fractional derivatives of these singular (generalized) functions are also defined by (2.10) with $p = -1$ and $-\frac{1}{2}$ (Gel'fand & Shilov 1964, pp. 115–122). These asymptotic relations are used in determining the full solution f_1 of (2.4) to obtain b_0 and v_0.

Here it should be remarked that (2.21 *c*), which is the differential equation to determine the 'shock displacement due to diffusivity' τ_1, is derived without recourse to matched asymptotic expansions. For the specific example of the Burgers equation considered by Crighton & Scott (1979, section 3), that expression corresponds to (3.15) in their paper. In fact, this can be verified if their example is recast in the present formulation by taking $f = -\theta/X$ for $-X^{\frac{1}{2}} < \theta < 0$, and $f = 0$ for $\theta < -X^{\frac{1}{2}}$ and $0 < \theta$, so that $V_0 = X^{-\frac{1}{2}}$, $V_2 = -X^{-1}$, $v_0 = -\tau_1/X$, $B_0 = b_0 = 0$ and $\tau_1 \equiv -A$ for $\eta = \theta+X^{\frac{1}{2}}-\beta\tau_1$. Hence it is found that the present approach using generalized functions can provide information on the shock displacement due to diffusivity.

2.2.2. *Inner expansion*

We have seen from (2.17) that the regular expansion in the outer region breaks down as η tends to zero. To remedy this non-uniformity, a shock layer is introduced so that the second-order derivative may now balance the nonlinear term just as in the Burgers equation. To describe this balance properly, we introduce instead of θ (but with X unchanged) a new coordinate ζ defined by $\zeta = [\theta - \tau(X)]/\beta = \eta/\beta$. In rewriting (1.1) in terms of ζ and X, the fractional derivative consists of two contributions, one from the rapid variation in the shock layer and the other from the slow variation in the outer region. Thus it is then expressed as

$$\int_{-\infty}^{\theta} \frac{1}{(\theta-\theta')^{\frac{1}{2}}}\frac{\partial f}{\partial \theta'}\mathrm{d}\theta' = \int_{-\infty}^{\tau-\Delta} \frac{1}{(\theta-\theta')^{\frac{1}{2}}}\frac{\partial f}{\partial \theta'}\mathrm{d}\theta' + \frac{1}{\beta^{\frac{1}{2}}}\int_{-\Delta/\beta}^{\zeta} \frac{1}{(\zeta-\zeta')^{\frac{1}{2}}}\frac{\partial f}{\partial \zeta'}\mathrm{d}\zeta'$$

$$= \int_{-\infty}^{0} \frac{1}{(-\eta')^{\frac{1}{2}}}\frac{\partial B}{\partial \eta'}\mathrm{d}\eta' + \frac{1}{\beta^{\frac{1}{2}}}\int_{-\infty}^{\zeta} \frac{1}{(\zeta-\zeta')^{\frac{1}{2}}}\frac{\partial f}{\partial \zeta'}\mathrm{d}\zeta' + o(1), \quad (2.22)$$

where $\tau - \Delta (0 < \beta \ll \Delta \ll 1)$ signifies a point in the so-called matching region located between the shock layer and the outer region. If we let the order of Δ be μ, then this satisfies $\mu/\beta \to \infty$ asymptotically as $\beta \to 0$.

With (2.22), (1.1) is now expressed as

$$\frac{\partial^2 \bar{f}}{\partial \zeta^2} + (\dot{\tau} + \bar{f})\frac{\partial \bar{f}}{\partial \zeta} = \beta^{\frac{1}{2}}\delta\frac{\partial^{\frac{1}{2}}\bar{f}}{\partial \zeta^{\frac{1}{2}}} + \beta\left(\frac{\partial \bar{f}}{\partial X} + \delta\frac{\partial^{\frac{1}{2}}B}{\partial \eta^{\frac{1}{2}}}\bigg|_{\eta=0}\right), \quad (2.23)$$

where $\bar{f}[=\bar{f}(\zeta, X)]$ denotes f in the shock layer.

According to the principle of matched asymptotic expansions, $\bar{f}$ should be determined so that it may be continued smoothly as $\zeta \to \pm\infty$ into the outer expansion as $\eta \to \pm 0$ through the matching region. The outer expansion to be matched is derived by expanding (2.9) and (2.17) around $\eta = 0$ and replacing η with $\beta\zeta$. It then follows that the outer expansion as $\zeta \to -\infty$ becomes, up to $O(\beta)$:

$$\bar{f}_{-\infty} = B_0 + \beta(b_0 + B_2\zeta) + \ldots, \quad (2.24)$$

while as $\zeta \to \infty$ it becomes

$$\bar{f}_{\infty} = B_0 + V_0 + \beta^{\frac{1}{2}}(V_1\zeta^{\frac{1}{2}} + v_{-1}\zeta^{-\frac{1}{2}}) + \beta[b_0 + v_0 + (B_2 + V_2)\zeta] + \ldots. \quad (2.25)$$

But because the matching should be executed over the intermediate matching region, the rigorous conditions require that

$$\lim_{\beta \to 0}\frac{1}{\beta}[\bar{f}_{\pm\infty}(\zeta, X;\beta) - \bar{f}(\zeta, X;\beta)] = 0, \quad (2.26)$$

where ζ is replaced by $(\mu/\beta)\chi$ with $\chi(\sim O(1))$ fixed (see e.g. Cole 1968). Here the right- and the left-hand matching regions correspond to $\chi > 0$ and $\chi < 0$, respectively.

We now seek $\bar{f}$ in the expanded form in powers of $\beta^{\frac{1}{2}}$:

$$\bar{f} = \bar{f}_0 + \beta^{\frac{1}{2}}\bar{f}_1 + \beta\bar{f}_2 + \ldots. \quad (2.27)$$

The lowest-order problem yields

$$\frac{\partial^2 \bar{f}_0}{\partial \zeta^2} + (\dot{\tau}_0 + \bar{f}_0)\frac{\partial \bar{f}_0}{\partial \zeta} = 0. \quad (2.28)$$

Imposing the matching conditions, it is found that $\dot{\tau}_0$ must satisfy (2.13), so that $\bar{f}_0$ is given by the well-known Taylor shock profile:

$$\bar{f}_0 = \tfrac{1}{2}(2B_0 + V_0 + V_0 \tanh Z), \tag{2.29}$$

where $Z = \frac{1}{4}V_0[\zeta - \zeta_0(X)]$.

The $O(\beta^{\frac{1}{2}})$-problem for $\bar{f}_1$ is given by

$$\frac{\partial \bar{f}_1}{\partial \zeta} + (\tfrac{1}{2}V_0 \tanh Z)\bar{f}_1 = \delta\left(\frac{V_0}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{Z} \frac{1}{(Z-Z')^{\frac{1}{2}}}(1+\tanh Z')\,\mathrm{d}Z', \tag{2.30}$$

where (2.29) and the matching condition as $\zeta \to -\infty$ have been used. The solution is easily obtained, by the standard procedure of variation of constants, as

$$\bar{f}_1 = \delta C \operatorname{sech}^2 Z,$$

with
$$C = \frac{4}{(\pi V_0)^{\frac{1}{2}}} \int_0^Z \mathrm{d}Z'' \cosh^2 Z'' \int_{-\infty}^{Z''} \frac{1}{(Z''-Z')^{\frac{1}{2}}}(1+\tanh Z')\,\mathrm{d}Z' + D, \tag{2.31}$$

where D is an arbitrary function of X.

Let us examine its asymptotic behaviour as $\zeta \to \infty$. (The asymptotic behaviour of $\bar{f}_1$ as $\zeta \to -\infty$ is given by $\bar{f}_1 \sim \delta(2V_0)^{\frac{1}{2}}\zeta \exp[\frac{1}{2}V_0(\zeta-\zeta_0)]$.) To evaluate the integral in C, $1+\tanh Z'$ is split into a sum of $2h(Z')$ and $1-2h(Z')+\tanh Z'$. It is found that the leading asymptotic behaviour results solely from $2h(Z')$, to give

$$\bar{f}_1 = \frac{4\delta}{\pi^{\frac{1}{2}}}\left[\zeta^{\frac{1}{2}} - \left(\tfrac{1}{2}\zeta_0 + \frac{1}{V_0}\right)\zeta^{-\frac{1}{2}} + \ldots\right] \quad \text{as} \quad \zeta \to \infty. \tag{2.32}$$

Incidentally, the contribution to C from the remaining term $1-2h(Z')+\tanh Z'$ is found to be small, of $O(\zeta^{-\frac{3}{2}})$. In the light of the matching condition as $\zeta \to \infty$, the leading term already corresponds exactly to the one in (2.25) (see (2.15) for V_1). The next term, proportional to $\zeta^{-\frac{1}{2}}$, seems to be unnecessary to match because it decays as $\zeta \to \infty$. But the matching condition (2.26) requires that this term *should* be matched, so that ζ_0 must be chosen to be $2/V_0$. This is the shock displacement in the presence of the hereditary effect. But since the location of the discontinuity $\tau(X)$ itself is also subject to the shift $\beta\tau_1(X)$ from τ_0, both ζ_0 and $\beta\tau_1$ contribute to the total shock displacement due to diffusivity.

Proceeding to the $O(\beta)$ problem, $\bar{f}_2$ is governed by

$$\frac{\partial^2 \bar{f}_2}{\partial \zeta^2} + \frac{\partial}{\partial \zeta}[(\tfrac{1}{2}V_0 \tanh Z)\bar{f}_2] = \frac{\partial \bar{f}_0}{\partial X} + \delta \left.\frac{\partial^{\frac{1}{2}} B}{\partial \eta^{\frac{1}{2}}}\right|_{\eta=0} - \dot{\tau}_1 \frac{\partial \bar{f}_0}{\partial \zeta} - \frac{\partial}{\partial \zeta}\left(\frac{\bar{f}_1^{\,2}}{2}\right) + \delta \frac{\partial^{\frac{1}{2}} \bar{f}_1}{\partial \zeta^{\frac{1}{2}}}. \tag{2.33}$$

This solution is sought by the same method as for $\bar{f}_1$, though it is straightforward but tedious, and found as

$$\begin{aligned} \bar{f}_2 = {} & -\frac{2}{V_0}\left(B_2 - \frac{\dot{V}_0}{V_0}\right)(2Z\tanh Z - 1 + Z^2 \operatorname{sech}^2 Z) + \frac{4\dot{V}_0}{V_0^2}Z - \dot{\zeta}_0 - \dot{\tau}_1 \\ & + \frac{4}{V_0}(\operatorname{sech}^2 Z)\int_{-\infty}^{Z}\left(-\tfrac{1}{2}\bar{f}_1^{\,2} + \delta\frac{\partial^{-\frac{1}{2}}\bar{f}_1}{\partial \zeta^{-\frac{1}{2}}}\right)\cosh^2 Z'\,\mathrm{d}Z' \\ & + E(\tanh Z + Z\operatorname{sech}^2 Z) + G\operatorname{sech}^2 Z, \end{aligned} \tag{2.34}$$

where B_2 is given in (2.14) and E and G are as yet unknown functions of X; for the definition of the fractional derivative of order $-\frac{1}{2}$, see (2.38) below. Taking the limit as $\zeta \to -\infty$, $\bar{f}_2$ is asymptotically given by

$$\bar{f}_2 = B_2\left(\frac{4}{V_0}Z\right) + \frac{2}{V_0}\left(B_2 - \frac{\dot{V}_0}{V_0}\right) - \dot{\zeta}_0 - \dot{\tau}_1 - E. \tag{2.35}$$

In the other limit as $\zeta \to \infty$, $\bar{f}_2$ is given by

$$\bar{f}_2 = (B_2 + V_2)\left(\frac{4}{V_0}Z\right) + \frac{2}{V_0}\left(B_2 - \frac{\dot{V}_0}{V_0}\right) - \dot{\zeta}_0 - \dot{\tau}_1 + E - 16\frac{\delta^2}{V_0^2}\left(1 - \frac{4}{\pi}\right). \tag{2.36}$$

Here V_2 is given in (2.15) and the integral in (2.34) has been evaluated as

$$\frac{4}{V_0}(\operatorname{sech}^2 Z)\int_{-\infty}^{Z}\left(-\tfrac{1}{2}\bar{f}_1^{\,2} + \delta\frac{\partial^{-\frac{1}{2}}\bar{f}_1}{\partial\zeta^{-\frac{1}{2}}}\right)\cosh^2 Z'\,\mathrm{d}z' = 16\frac{\delta^2}{V_0^2}\left(1 - \frac{4}{\pi}\right)(Z-1) + \ldots, \tag{2.37}$$

where (2.32) and the following asymptotic expression are used:

$$\frac{\partial^{-\frac{1}{2}}\bar{f}_1}{\partial\zeta^{-\frac{1}{2}}} \equiv \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\zeta}\frac{\bar{f}_1(\zeta')}{(\zeta-\zeta')^{\frac{1}{2}}}\mathrm{d}\zeta' = 2\delta\left(\zeta - \zeta_0 - \frac{2}{V_0}\right) + \ldots \quad \text{as} \quad \zeta \to \infty. \tag{2.38}$$

In (2.35) and (2.36), it is immediately found that both the coefficients of ζ $(4Z/V_0 = \zeta - \zeta_0)$ agree automatically with those of ζ in (2.24) and (2.25). The remaining matching conditions for the constant term in ζ yield two conditions for $\dot{\tau}_1$ and E:

$$\begin{aligned}\dot{\tau}_1 &= -b_0 - \tfrac{1}{2}v_0 - \dot{\zeta}_0 - (B_2 + \tfrac{1}{2}V_2)\,\zeta_0 + \frac{2}{V_0}\left[B_2 - \frac{\dot{V}_0}{V_0} - 4\frac{\delta^2}{V_0}\left(1 - \frac{4}{\pi}\right)\right] \\ &= -b_0 - \tfrac{1}{2}v_0 - \frac{2}{V_0^2}\left[2\dot{B}_0 + \dot{V}_0 + 2\delta\left.\frac{\partial^{\frac{1}{2}}B}{\partial\eta^{\frac{1}{2}}}\right|_{\eta=0} + 6\delta^2\left(1 - \frac{4}{\pi}\right)\right],\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}E &= \tfrac{1}{2}v_0 + \tfrac{1}{2}V_2\,\zeta_0 + 8\frac{\delta^2}{V_0^2}\left(1 - \frac{4}{\pi}\right) \\ &= \tfrac{1}{2}v_0 + \frac{2}{V_0^2}\left[2\dot{B}_0 + \dot{V}_0 + 2\delta\left.\frac{\partial^{\frac{1}{2}}B}{\partial\eta^{\frac{1}{2}}}\right|_{\eta=0} + 6\delta^2\left(1 - \frac{4}{\pi}\right)\right],\end{aligned} \tag{2.40}$$

where the final expressions for $\dot{\tau}_1$ and E are derived for $\zeta_0 = 2/V_0$. It is found that (2.39) agrees with (2.21 *c*). For the Burgers equation, if $\dot{\tau}_1$ satisfies (2.21 *c*), ζ_0 is found from the first relation in (2.39) to have an arbitrariness proportional to V_0^{-1}. But there arises no need to take account of it in addition to τ_1. The remaining D and G in (2.31) and (2.34), which are the coefficients of the homogeneous solutions to (2.30), have not yet been determined by the matching. Both D and G will contribute to the higher-order shock displacement at order $\beta^{\frac{3}{2}}$ and β^2, respectively, because those terms can be incorporated in (2.27) into a modified argument of $\tanh Z$ in $\bar{f}_0$. Hence we have matched the shock-layer solutions with the outer expansion up to order β inclusive.

2.3. *The evolution of the discontinuity*

The discussions in the preceding sections are based on the *a priori* assumption that the discontinuity already exists, though smoothed through the shock layer. In this section, we examine how the discontinuity emerges in the course of evolution, and then how it eventually decays. Furthermore we preview the overall evolution from the standpoint of the asymptotic analysis.

In solving (2.2), there exists a 'shock-formation point', $\theta = \theta_0$ and $X = X_0$, at which the waveform begins to be multi-valued. For the Burgers equation, Crighton & Scott (1979) introduced the idea of an 'embryo-shock region' around the shock formation point, through which the local dissipation and nonlinearity balance transiently to form eventually a fully developed Taylor shock profile. Following their idea, we examine a similar embryo-shock region for the present problem. When f_0 in (2.2) does not vanish at the shock formation point, we magnify θ and X around θ_0 and X_0 by $(\theta-\theta_0)/\beta = \zeta_0$ and $(X-X_0)/\beta = \sigma$. In terms of ζ_0 and σ, (1.1) is reduced to the Burgers equation at leading order:

$$\frac{\partial f}{\partial \sigma} - f\frac{\partial f}{\partial \zeta_0} = \frac{\partial^2 f}{\partial \zeta_0^2} + o(1). \tag{2.41}$$

In the case when $f_0(\zeta_0, X_0)$ vanishes, we adopt different scalings, $(\zeta-\zeta_0)/\beta^{\frac{3}{4}} = \zeta_0$ and $(X-X_0)/\beta^{\frac{1}{2}} = \sigma$, so that $f/\beta^{\frac{1}{4}}$ obeys (2.41) at the leading order. In either event, it is found that the hereditary effect is of higher order. Thus we must solve this transient Burgers equation first by taking the regular expansion just before the shock formation point as an initial condition so that its solution may be matched with the fully developed shock wave. For the details, see Crighton & Scott (1979).

Next we examine the opposite case in which the discontinuity in f_0 vanishes at a finite value of X. Then how will the discontinuity evolve after that? For simplicity, we examine a case in which $B(\eta, X)$ vanishes identically. It has already been remarked that (2.9) exhibits a non-uniformity as V_0 vanishes. Consider a point $\theta = \theta_1$ and $X = X_1$ at which V_0 may be regarded as infinitesimally small. The further evolution around the wavefront can be examined by linearizing (2.2) locally as

$$\frac{\partial f_0}{\partial X} = -\delta \frac{\partial^{\frac{1}{2}} f_0}{\partial \theta^{\frac{1}{2}}}. \tag{2.42}$$

The 'initial' condition at $X = X_1$ is then taken locally around the wavefront as

$$f_0(\theta, X = X_1) = V_0 h(\theta - \theta_1), \tag{2.43}$$

where V_0 is infinitesimally small. To solve (2.42) subject to (2.43), we make use of the Fourier transform method. Then f_0 is easily obtained as

$$f_0 = \begin{cases} V_0\left[1 - \dfrac{2}{\pi^{\frac{1}{2}}}\displaystyle\int_0^Y \exp(-y^2)\,\mathrm{d}y\right] & (0 < \theta - \theta_1 \ll 1), \\ 0 & (\theta - \theta_1 < 0), \end{cases} \tag{2.44}$$

where $Y = \delta(X-X_1)/[4(\theta-\theta_1)]^{\frac{1}{2}}$. It is found from this that any derivatives of f_0 (of integral order) with respect to θ vanish at $\theta = \theta_1$, because the first-order derivative is given by

$$\frac{\partial f_0}{\partial \theta} = \frac{\delta V_0 (X-X_1)}{[4\pi(\theta-\theta_1)^3]^{\frac{1}{2}}} \exp\left[-\frac{\delta^2 (X-X_1)^2}{4(\theta-\theta_1)}\right]. \tag{2.45}$$

This suggests that as soon as V_0 vanishes at $X = X_1 + 0$, any discontinuity at $\theta = \theta_1$ disappears instantaneously for $X > X_1$, and the region $\theta < \theta_1$ ahead of the signal remains undisturbed. This is the origin of the non-uniformity of (2.9) as $V_0 \to 0$. Since θ denotes physically the retarded time in a frame moving with the linear sound speed, the wavefront cannot be propagated faster than the sound speed so that the wave tends to relax only in the region behind. The further evolution is well described by (2.2), if no other discontinuity exists. In view of (2.45), incidentally, the linearization around the wavefront is justified because the ratio $|(f_0 \partial f_0/\partial\theta)/(\partial f_0/\partial X)|$ vanishes as $|f_0(X-X_1)/(\theta-\theta_1)|$ when θ tends to θ_1.

So far, we have not imposed any restriction on the magnitude of δ. But it is usually the case that δ is relatively smaller than unity, though $\delta \gg \beta$ (Sugimoto 1990). In that case, we can anticipate the following scenario for the evolution. In the near field $X \lesssim O(1)$, (1.1) is governed mainly by the lossless nonlinear hyperbolic wave equation on the left-hand side, and its solution is generally subject to nonlinear steepening. As it becomes steep enough, the second-order derivative now comes into play quickly, though only very locally. Its effect begins with the embryo-shock region, and leads eventually to the Taylor shock profile. This process is well described by the Burgers equation. In the outer region of the shock layer, the lossless solution remains valid. As $X \gg 1$, f is approximated by $f = -[\theta - \theta_0(X)]/X$, where $\theta_0(X)$ ($|\mathrm{d}\theta_0/\mathrm{d}X| \ll |\theta-\theta_0|/X$) is a function of X. Hence the global profile of f evolves into a series of triangular pulses, in general, if the small effect of δ is ignored.

But X becomes large in the far field, comparable with δ^{-2}, and so the hereditary effect now comes into play. For the slow and long scales $\bar{\theta}$ and $\bar{X}$ defined by $\delta^2\theta$ and $\delta^2 X$, respectively, f is governed by

$$\frac{\partial f}{\partial \bar{X}} - f\frac{\partial f}{\partial \bar{\theta}} = -\frac{\partial^{\frac{1}{2}} f}{\partial \bar{\theta}^{\frac{1}{2}}} + O(\beta\delta^2). \tag{2.46}$$

A transition from the Burgers region to this hereditary far field should be determined by the matching principle as $X \rightarrow \infty$ but $\bar{X} \rightarrow 0$.

In the far field, it often happens that f decays. Then the nonlinear term in (2.46) becomes so small that the evolution is well described by the linearized equation. This equation is simply the counterpart of the heat equation factorized†:

$$\left(\frac{\partial}{\partial \bar{X}} - \frac{\partial^{\frac{1}{2}}}{\partial \bar{\theta}^{\frac{1}{2}}}\right)\left(\frac{\partial}{\partial \bar{X}} + \frac{\partial^{\frac{1}{2}}}{\partial \bar{\theta}^{\frac{1}{2}}}\right) f = \frac{\partial^2 f}{\partial \bar{X}^2} - \frac{\partial f}{\partial \bar{\theta}} = 0. \tag{2.47}$$

Of course, however, there exists a case in which the nonlinearity survives, as in the evolution from the positive step. In this case, we have immediately an embryo-shock region around $X = \theta = 0$, through which the Taylor shock profile is formed for $X \gg \beta$. But its magnitude is still preserved at order unity in the far field $X \sim O(\delta^{-2})$, while its profile is now regarded asymptotically as the step function in the far-field variable $\bar{\theta}$. Further evolution in the far-field variables $\bar{\theta}$ and $\bar{X}$ is simply that of the positive step demonstrated by Sugimoto (1990).

3. Numerical analysis

3.1. *Finite-difference method*

In the light of the results of the asymptotic analysis, let us now solve (1.1) numerically subject to (1.2) to confirm the evolution scenario. For numerical calculations, it is interesting, and also usually the case, to take δ small compared with unity ($\beta \ll \delta \ll 1$), so that a shock will emerge. In this section, we employ the finite-difference method (see, for example, Mitchell & Griffiths 1980) to solve the evolution from localized initial conditions.

In the lossless and Burgers regions, we use the implicit finite-difference method due to Lee-Bapty & Crighton (1987). For a 3×3 grid in (θ, X)-space, the derivative with respect to X is approximated by the central difference, while the derivatives with

† It is interesting to compare this factorization with those for the second-order hyperbolic wave equation $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = (\partial/\partial t + \partial/\partial x)(\partial u/\partial t - \partial u/\partial x) = 0$ and for the elliptic Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (\partial/\partial x + \mathrm{i}\,\partial/\partial y)(\partial u/\partial x - \mathrm{i}\,\partial u/\partial y) = 0$ which suggest the introduction of the characteristic variables $\varphi = x - t$ and $\psi = x + t$, and the complex-conjugate pairs $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, respectively.

respect to θ are evaluated not simply by the central ones but by averaging over the three X-rows. But f in the nonlinear term and the half-order derivative are evaluated only at the central point of the 3×3 grid.

In evaluating the half-order derivative, the simple Simpson's rule and its modification are used. The integral (1.3) is divided into two parts:

$$\int_{-\infty}^{\theta}\frac{1}{(\theta-\theta')^{\frac{1}{2}}}\frac{\partial f}{\partial\theta'}\,\mathrm{d}\theta' = \left(\int_{-\infty}^{\theta-2\Delta\theta}+\int_{\theta-2\Delta\theta}^{\theta}\right)\frac{1}{(\theta-\theta')^{\frac{1}{2}}}\frac{\partial f}{\partial\theta'}\,\mathrm{d}\theta', \tag{3.1}$$

where $\Delta\theta$ $(0<\Delta\theta\leqslant 1)$ is taken small enough. Because of the weak singularity of the kernel function $(\theta-\theta')^{-\frac{1}{2}}$ at $\theta'=\theta$, the first integral on the right-hand side is evaluated by the usual Simpson's rule. The second integral is evaluated by approximating f by a quadratic function around $\theta'=\theta$, so that

$$\int_{\theta-2\Delta\theta}^{\theta}\frac{1}{(\theta-\theta')^{\frac{1}{2}}}\frac{\partial f}{\partial\theta'}\,\mathrm{d}\theta' = \left(\frac{2}{9\Delta\theta}\right)^{\frac{1}{2}}[-7f(\theta-\Delta\theta)+8f(\theta)-f(\theta+\Delta\theta)]+O[(\Delta\theta)^{\frac{5}{2}}], \tag{3.2}$$

where the dependence on X is suppressed. Because the infinite lower limit in (3.1) must be replaced by some finite value $\theta=-L<0$, it happens that the above recipe cannot be applied when $\theta+L$ is an odd multiple of $\Delta\theta$. Then the trapezoidal rule is used only for the end interval. In particular, at $\theta=-L+\Delta\theta$ the integration is made by approximating f by a straight line.

In the calculations, we set the upper bound of θ at $\theta=M>0$, as well as the lower one at $\theta=-L$, to lie within a finite region. As will be seen later, the half-order derivative generates a long tail in the course of evolution which extends past the upper bound at $\theta=M$. At this end point, therefore, the condition $\partial^2 f/\partial\theta^2=0$ is imposed, which releases the tail out of the region. At the other end, $\theta=-L$, the condition $\partial f/\partial\theta=0$ is imposed, which, however, is not significant in the results. Rather, for the condition at $\theta=M$ there may arise the question as to how to check the validity of the results by a conservation law. This problem will be considered in §4.

Up to the point where the Burgers region is attained, the implicit method is very stable and successful. But this method is very time-consuming and, even worse, leads to accumulation of errors in the far field. In order to pursue such a long-time behaviour, the explicit method is rather suitable. For a 3×2 grid in (θ, X)-space, the derivative with respect to X is approximated by the forward difference, while the derivatives with respect to θ are evaluated, as before, by the central differences but in the current X-row only. In evaluating the nonlinear term, f is now averaged over the current three grid points, but the half-order derivative is evaluated at the central point only. The timing of the changeover from the implicit scheme to the explicit one is set when the Burgers region is attained. Incidentally, if δ has such a large value as unity, shown in figures 1 and 2, the implicit scheme is unstable and the explicit one is advisable from the outset.

In the following, we consider four typical cases of the initial condition $F(\theta)$:

$$\left.\begin{aligned} &\text{(I)} & F(\theta) &= \exp(-\theta^2),\\ &\text{(II)} & F(\theta) &= -\exp(-\theta^2),\\ &\text{(III)} & F(\theta) &= -(2\mathrm{e})^{\frac{1}{2}}\theta\exp(-\theta^2),\\ &\text{(IV)} & F(\theta) &= (2\mathrm{e})^{\frac{1}{2}}\theta\exp(-\theta^2),\end{aligned}\right\} \tag{3.3}$$

where the maximum of F is normalized to unity. For the Burgers equation, the

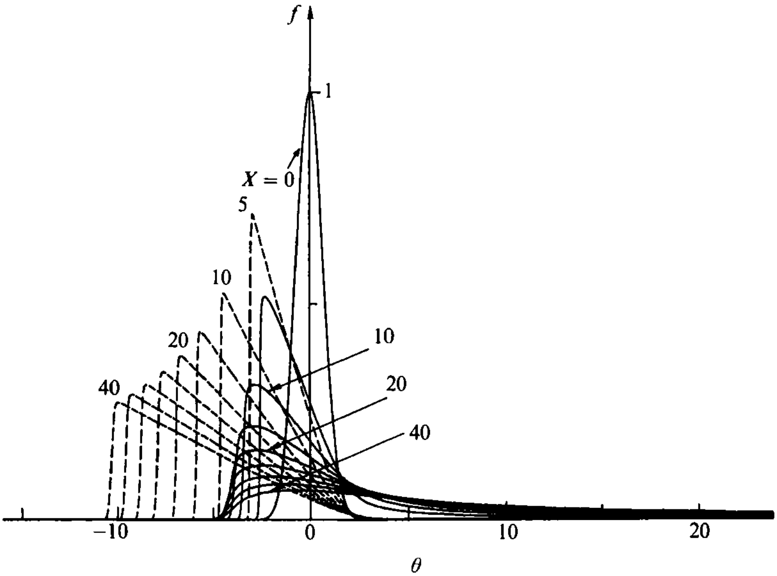

FIGURE 3. Evolution from the initial condition (I): $F(\theta) = \exp(-\theta^2)$; the solid lines represent the evolution of f by (1.1) with $\beta = 0.01$ and $\delta = 0.1$ up to $X = 40$ by steps in X of 5, while the broken lines represent that by the Burgers equation with $\beta = 0.01$ (and $\delta = 0$).

evolution from condition (II) can be reproduced from (I) by transforming f to $-f$ and θ to $-\theta$. When the half-order derivative is present, however, condition (II) is no longer reducible.

For each initial condition, figures 3–6 display the evolution with respect to X, where the broken lines represent, for reference, the evolution of the Burgers equation ((1.1) with the same value of β, but $\delta = 0$) under the same condition. To demonstrate the results of the asymptotic analysis, it is preferable to take β as small as possible with the inequality $\beta \ll \delta$ preserved. In view of the time consumed, however, β and δ are chosen, respectively, to be 10^{-2} and 10^{-1}. The grid sizes $\Delta\theta$ and ΔX are chosen to be $\Delta\theta = 2.5 \times 10^{-2}$ and $\Delta X = 10^{-2}$, except for (IV) where the size of $\Delta\theta$ is halved. The calculation region $[-L, M]$ is taken wide enough, as $[-8, 48]$, $[-4, 52]$, $[-8, 48]$ and $[-4, 32]$ for (I) to (IV), respectively. In figures 3–6, however, the profiles are not displayed over the full ranges of the above calculation regions.

As suggested by the asymptotic analysis, the hereditary effect does not manifest itself so remarkably up to $X = 5$ in figures 3 to 5, and up to $X = 2$ in figure 6, as long as errors of $O(\delta)$ are ignored. Incidentally, the changeover from the implicit scheme to the explicit one is made at $X = 5$ for all cases. In figure 3, the waveform at $X = 5$ may well be regarded as a triangular pulse with discontinuity of about 0.5. This should be compared with the solution of (2.2) demonstrated by Sugimoto (1990). The discontinuity is now replaced by a sharp shock layer. As (2.9) suggests, the waveform at $X = 10$ shows the discontinuity followed by the square-root function, so that the wavefront appears to be rounded. At the final stage $X = 40$, it is seen that the discontinuity has disappeared. The wavefront appears to be smooth enough with respect to θ, as (2.45) predicts, and it is no longer propagated forward (leftward). Next, figure 4 shows the interesting case of the negative pulse. Owing to the hereditary effect, it is impossible for the negative discontinuity to be propagated with the equilibrium state $f = 0$ on the positive side of θ. Therefore even if it were applied initially (like the Burgers solution at $X = 5$), it spreads instantaneously

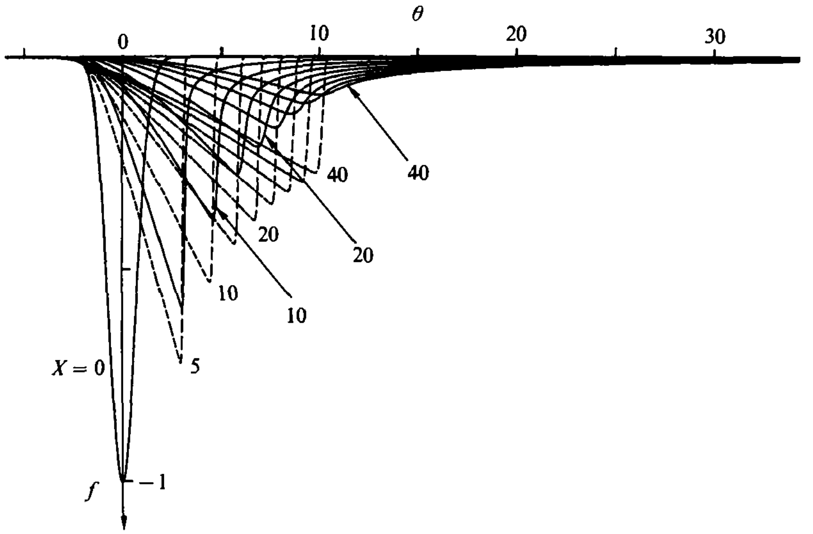

FIGURE 4. Evolution from the initial condition (II): $F(\theta) = -\exp(-\theta^2)$; the solid lines represent the evolution of f by (1.1) with $\beta = 0.01$ and $\delta = 0.1$ up to $X = 40$ by steps in X of 5, while the broken lines represent that by the Burgers equation with $\beta = 0.01$ (and $\delta = 0$).

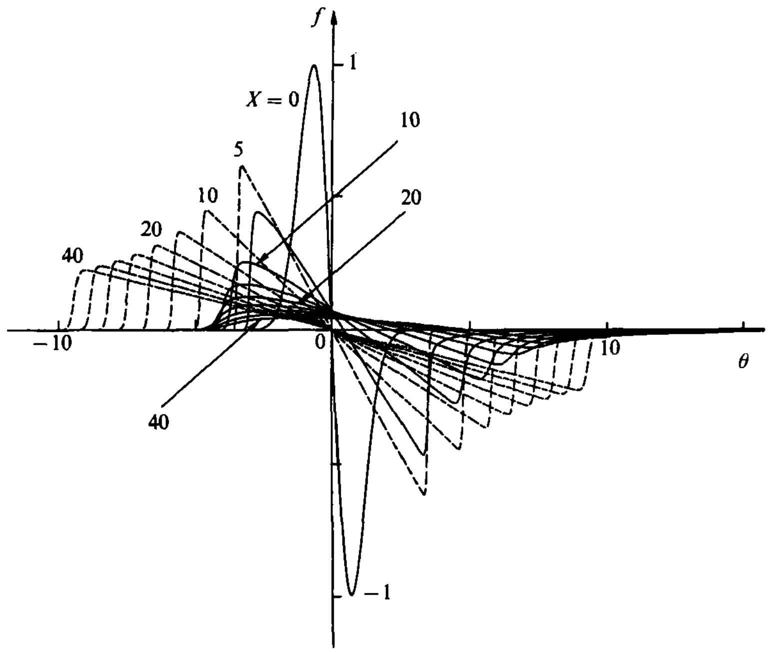

FIGURE 5. Evolution from the initial condition (III): $F(\theta) = -(2e)^{\frac{1}{2}}\theta\exp(-\theta^2)$; the solid lines represent the evolution f by (1.1) with $\beta = 0.01$ and $\delta = 0.1$ up to $X = 40$ by steps in X of 5, while the broken lines represent that by the Burgers equation with $\beta = 0.01$ (and $\delta = 0$).

toward positive θ. Figure 5 is the combined case of figures 3 and 4. In the Burgers solutions, the positive pulse and the negative pulse behave antisymmetrically with respect to the origin. With the half-order derivative, interestingly, the effect of the positive pulse is carried over to strengthen the negative one slightly. The opposite-polarity case of this 'merging' is shown in figure 6.

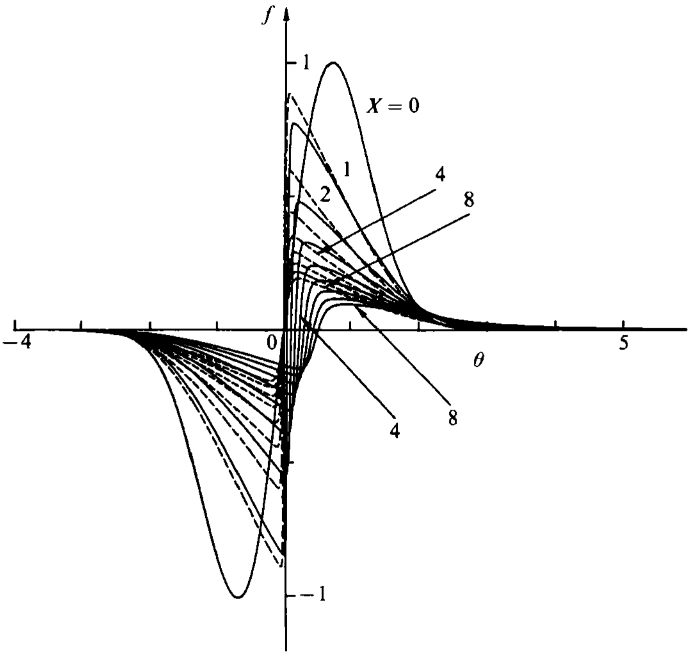

FIGURE 6. Evolution from the initial condition (IV): $F(\theta) = (2e)^{\frac{1}{2}}\theta\exp(-\theta^2)$; the solid lines represent the evolution of f by (1.1) with $\beta = 0.01$ and $\delta = 0.1$ up to $X = 8$ by steps in X of 1, while the broken lines represent that by the Burgers equation with $\beta = 0.01$ (and $\delta = 0$).

In the far field, generally speaking, the hereditary effect decelerates the propagation, compared with the Burgers solutions, and gives rise to pronounced overall dissipation and distortion of waveform (lengthened backward). The former phase lag is due partly to the linear dispersion involved in the half-order derivative. For the sinusoidal wave $f \propto \exp[\mathrm{i}(\omega\theta - KX)]$, the linear dispersion relation of (1.1) is given by $-\mathrm{i}K = -\beta\omega^2 - \delta(\mathrm{i}\omega)^{\frac{1}{2}}$ so that f becomes

$$f \propto \exp\{-[\beta\omega^2 + \delta(\tfrac{1}{2}\omega)^{\frac{1}{2}}]X\}\exp\{\mathrm{i}\omega[\theta - \delta(2\omega)^{-\frac{1}{2}}X]\}, \tag{3.4}$$

where $\omega\,(>0)$ and K denote a frequency and a wavenumber. Thus the phase moves with speed $\delta(2\omega)^{-\frac{1}{2}}$ toward the positive direction of θ. This explains why the locations of the positive peaks lag behind. Interestingly enough, though, the locations of the negative peaks in figures 4 and 5 coincide with those in the Burgers solutions. As for the profiles, on the other hand, the sharp shock layer disappears in the far field, while the waveform is lengthened behind to form the tail. The appearance of the tail is a typical feature of the hereditary effect. Even if the profile is localized initially, it is globally distributed eventually. Hence it is found that the hereditary effect exhibits a sharp contrast to the diffusive effect.

3.2. *Spectral method*

Next we shall examine the periodic evolution in θ from a sinusoidal initial condition by the spectral (Galerkin) method (see Gazdag 1973; Basdevant *et al.* 1986; Sachdev 1987). Taking the 2π-period in $[-\pi, \pi]$, f is discretized at N (even) points $\theta_j = 2\pi j/N$ ($j = -\frac{1}{2}N, -\frac{1}{2}N+1, \ldots, \frac{1}{2}N-1$) by a finite Fourier series:

$$f(\theta_j, X) = \sum_{k=-N/2}^{N/2-1} C_k(X)\exp(\mathrm{i}k\theta_j), \tag{3.5}$$

where the complex spectra $C_k(X)$ $(k = -\frac{1}{2}N, -\frac{1}{2}N+1, \ldots, \frac{1}{2}N-1)$ are inversely expressed by

$$C_k(X) = \frac{1}{N}\sum_{j=-N/2}^{N/2-1} f(\theta_j, X)\exp(-\mathrm{i}k\theta_j). \tag{3.6}$$

Here C_k satisfy the conditions $C_{-k} = C_k^*$ where the asterisk denotes the complex conjugate. Then the derivatives of f, including those of fractional order p, are simply calculated by

$$\frac{\partial^p f}{\partial\theta^p}(\theta_j, X) = \sum_{k=-N/2}^{N/2-1} (\mathrm{i}k)^p C_k \exp(\mathrm{i}k\theta_j), \tag{3.7}$$

where p takes the values of 1, 2 and $\frac{1}{2}$. For $p = \frac{1}{2}$, the formula is defined as the Fourier transform of the algebraic generalized function (Sugimoto 1989), although it is in fact simply the classically well-known Fresnel integral.

For the product of f with any function g whose spectra are given by D_k, the Fourier coefficients E_k of the product fg are calculated by

$$E_k = \sum_{k'=-N/2}^{N/2-1} C_{k'} D_{k-k'}, \tag{3.8}$$

where D_k are defined outside the original range of suffices $(-\frac{1}{2}N \leqslant k \leqslant \frac{1}{2}N-1)$ by $D_{k\pm N} = D_k$.

Following the procedure developed by Gazdag (1973) (see also Sachdev 1987), the evolution of f from X by ΔX $(\ll 1)$ is calculated by a Taylor expansion taken up to the third order in ΔX:

$$f(X+\Delta X) = f(X) + \frac{\partial f}{\partial X}\Delta X + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(\Delta X)^2 + \frac{1}{6}\frac{\partial^3 f}{\partial X^3}(\Delta X)^3 + O[(\Delta X)^4], \tag{3.9}$$

where the right-hand side is evaluated at X, the dependence on θ being suppressed. Here the derivatives with respect to X are evaluated by differentiating the original equation (1.1):

$$\left.\begin{aligned} f_X &= ff_\theta + \beta f_{\theta\theta} - \delta f_{\theta^{\frac{1}{2}}}, \\ f_{XX} &= f_X f_\theta + ff_{X\theta} + \beta f_{X\theta\theta} - \delta f_{X\theta^{\frac{1}{2}}}, \\ f_{XXX} &= f_{XX} f_\theta + 2 f_X f_{X\theta} + ff_{XX\theta} + \beta f_{XX\theta\theta} - \delta f_{XX\theta^{\frac{1}{2}}}, \end{aligned}\right\} \tag{3.10}$$

where the suffixes designate partial differentiation with respect to θ and/or X, while $f_{\theta^{\frac{1}{2}}}$ denotes the half-order derivative of θ for simplicity. To evaluate (3.10) at X, the derivatives of X on the right-hand sides are rewritten in terms of the derivatives of θ only by using the lower-order derivatives of X. Introducing (3.10) thus expressed into (3.9) and using the relations (3.7) and (3.8), we derive the marching scheme with respect to X for each Fourier component.

According to this scheme, we solve (1.1) under the initial condition $F(\theta) = \sin\theta$, with $\beta = 10^{-2}$ and $\delta = 10^{-1}$, by taking $N = 256$ and $\Delta X = 2\times 10^{-3}$. The numerical results are shown in figure 7 together with the evolution under the Burgers equation. This sinusoidal case is seen to be qualitatively similar to figure 6 if such a pulse were applied repeatedly, with an appropriate period, from some large negative θ. But in the periodic case, the effect of the tail from the initial step-up disappears completely. In this case, the phase moves, by (3.4), with the speed $\delta(2\omega)^{-\frac{1}{2}}$ toward the positive direction of θ, where ω is unity initially. Because $\omega^{-\frac{1}{2}}$ is a relatively insensitive function of ω, however, the speed is almost constant, even if the higher harmonics

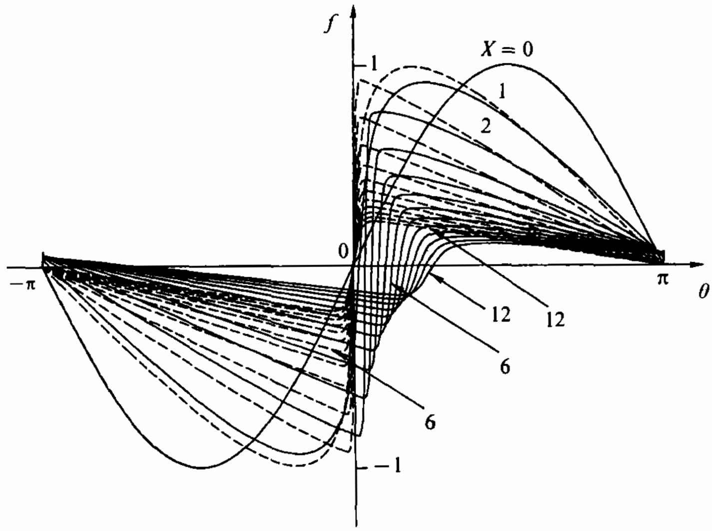

FIGURE 7. Evolution from the sinusoidal initial condition $F(\theta) = \sin\theta$; the solid lines represent the evolution of f by (1.1) with $\beta = 0.01$ and $\delta = 0.1$ up to $X = 12$ by steps in X of 1, while the broken lines represent that by the Burgers equation with $\beta = 0.01$ (and $\delta = 0$).

($\omega = 2, 3, 4, \ldots$) are generated by the nonlinearity. This seems to be the reason why the points at which $f = 0$ in figure 7 appear to lag at a constant rate in spite of the nonlinear distortion. In addition to the phase lag, the waveform tends to become a rounded N-wave ('S'-wave might be appropriate) and to form a hunchbacked shape. At the final stage of evolution, the sharp shock layer disappears, and the further evolution will be described by the linearized version of (2.46). Quite recently, Gittler & Kluwick (1989) have applied the spectral (Galerkin) method to a similar equation (asymptotically equivalent) to (1.1). Although δ is chosen relatively large, $\delta \sim \pi^{\frac{1}{2}}$, so that the calculation is limited to a short period of X, the initial behaviour of the present results appears to agree qualitatively with their results.

4. Conservation law

In this section, we mention the conservation law derived from (1.1), which is used to check the numerical results by the finite-difference method. For localized conditions satisfying $f \to 0$ and $\partial f/\partial\theta \to 0$ as $\theta \to \pm\infty$, integration of (1.1) over the whole range of θ yields

$$\frac{\mathrm{d}}{\mathrm{d}X}\int_{-\infty}^{\infty} f(\theta, X)\,\mathrm{d}\theta = -\delta\int_{-\infty}^{\infty}\frac{\partial^{\frac{1}{2}} f}{\partial\theta^{\frac{1}{2}}}\mathrm{d}\theta. \tag{4.1}$$

To evaluate the right-hand side, we use the following relations:

$$\int_{-\infty}^{\infty}\frac{\partial^{\frac{1}{2}} f}{\partial\theta^{\frac{1}{2}}}\mathrm{d}\theta = \lim_{\omega\to 0}\int_{-\infty}^{\infty}\frac{\partial^{\frac{1}{2}} f}{\partial\theta^{\frac{1}{2}}}\exp(\mathrm{i}\omega\theta)\,\mathrm{d}\theta = \lim_{\omega\to 0}(-2\pi\mathrm{i}\omega)^{\frac{1}{2}}\hat{f}(\omega, X), \tag{4.2}$$

where $\hat{f}$ stands for the Fourier transform of f, defined by

$$\hat{f}(\omega, X) \equiv \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(\theta, X)\exp(\mathrm{i}\omega\theta)\,\mathrm{d}\theta. \tag{4.3}$$

If $\hat{f}$ is bounded as $\omega \to 0$, i.e.

$$\lim_{\omega\to 0}\left|\int_{-\infty}^{\infty} f\exp(\mathrm{i}\omega\theta)\,\mathrm{d}\theta\right| = \left|\int_{-\infty}^{\infty} f\,\mathrm{d}\theta\right| < \infty, \tag{4.4}$$

then it is found that the right-hand side of (4.1) vanishes, so that the integral of f over the whole region of θ is conserved with respect to X. Incidentally, for periodic initial conditions as well, it is easy to show that the integral of f over one period is conserved. In fact, integration of (1.1) leads to

$$\frac{\partial}{\partial X}\int_{\theta}^{\theta+T} f(t,X)\,\mathrm{d}t = -\delta\frac{\partial^{-\frac{1}{2}}f}{\partial t^{-\frac{1}{2}}}\bigg|_{t=\theta}^{t=\theta+T} = -\frac{\delta}{\pi^{\frac{1}{2}}}\int_{-\infty}^{t}\frac{1}{(t-t')^{\frac{1}{2}}}f(t',X)\,\mathrm{d}t'\bigg|_{t=\theta}^{t=\theta+T} = 0, \tag{4.5}$$

where θ is arbitrary and T is a period of f, i.e. $f(\theta\pm T, X) = f(\theta, X)$.

By the conservation law, we can check the accuracy of the numerical results obtained by the finite-difference method. As was shown, the waveform tends to have a slowly decreasing tail, which extends out of the calculation region as X increases. Thus it is evident that the conservation law does not hold without taking account of the tail extent. For the initial condition (II) as the worst case, it is found that the quantity to be conserved is reduced to 70% of its initial value at the final stage of calculation if only the finite region $-4 \leqslant \theta \leqslant 52$ is considered. So there arises a need to complement this by taking account of the tail beyond the calculation region.

To this end, asymptotic solutions are developed for the tails by solving (2.46) in the far field. According to the matching principle, the initial condition for this region at $\bar{X} = 0$ is taken from the asymptotic solution to the Burgers equation as $X \to \infty$ but $\bar{X} \to 0$. Suppose this condition at $\bar{X} = 0$ be $f = W(\bar{\theta})$, where $W(\bar{\theta})$ depends also on an arbitrary matching constant. For localized initial conditions at $X = 0$, the asymptotic solutions to the Burgers equation decay rapidly enough to allow the linearization of (2.46). Then we can solve it easily by using the Fourier transform method, to get

$$\hat{f}(\omega,\bar{X}) = \hat{W}(\omega)\exp[-(-\mathrm{i}\omega)^{\frac{1}{2}}\bar{X}], \tag{4.6}$$

where $\hat{W}(\omega)$ denotes the Fourier transform of $W(\bar{\theta})$ (whose definition is given by (4.3)). Since we are concerned with the asymptotic behaviour as $\bar{\theta} \to \infty$ we expand $\hat{W}(\omega)$ around $\omega = 0$. This expansion is possible from the assumption that W is localized in $\bar{\theta}$. It then follows that

$$\hat{f} = [\hat{W}(0) + \mathrm{i}\frac{\mathrm{d}\hat{W}(0)}{\mathrm{d}\omega}(-\mathrm{i}\omega) + \dots]\exp[-(-\mathrm{i}\omega)^{\frac{1}{2}}\bar{X}]. \tag{4.7}$$

Using the formula of the inverse transform

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp[-(-\mathrm{i}\omega)^{\frac{1}{2}}\bar{X}]\exp(-\mathrm{i}\omega\bar{\theta})\,\mathrm{d}\omega = (1+\operatorname{sgn}\bar{\theta})\frac{\bar{X}}{(2|\bar{\theta}|)^{\frac{3}{2}}}\exp\left(-\frac{\bar{X}^2}{4|\bar{\theta}|}\right), \tag{4.8}$$

we have from (4.7)

$$f = \left[\hat{W}(0) + \mathrm{i}\frac{\mathrm{d}\hat{W}(0)}{\mathrm{d}\omega}\frac{\partial}{\partial\bar{\theta}} + \dots\right]\frac{\bar{X}}{\sqrt{2}\,\bar{\theta}^{\frac{3}{2}}}\exp\left(-\frac{\bar{X}^2}{4\bar{\theta}}\right) \quad \text{for} \quad \bar{\theta} > 0. \tag{4.9}$$

Here note that (4.8) is the elementary solution of the linearized equation (2.46). In view of this, the first term in (4.9) corresponds to the 'monopole' solution and the second term to the 'dipole' solution. Since $(2\pi)^{\frac{1}{2}}\hat{W}(0)$ represents the integral of $W(\bar{\theta})$, it is found that for $\hat{W}(0) \neq 0$, the tail extends farther than in the case with $\hat{W}(0) = 0$. For $\bar{\theta} \gg 1$, the tail decays algebraically with respect to $\bar{\theta}$, in contrast to the exponential decay in the Burgers equation.

Now we apply the above result to the specific cases. As far as the asymptotic solution to the Burgers equation as $X \to \infty$ is concerned, the initial condition (I) is equivalent to the delta function $F(\bar{\theta}) = \pi^{\frac{1}{2}}\delta(\bar{\theta})$ in the far-field variable (see (4.8) and (4.9) in Whitham 1974). It is then shown that the asymptotic form is given by a triangular shape in $-l < \bar{\theta} < 0$ with $l = \delta^2(2\pi^{\frac{1}{2}}X)^{\frac{1}{2}}$ and the transition layers are too thin to be 'visible'. Thus $W(\bar{\theta})$ is taken as

$$W(\bar{\theta}) = \begin{cases} -\dfrac{2\delta^2\pi^{\frac{1}{2}}\bar{\theta}}{l^2} & \text{for} \quad -l \leqslant \bar{\theta} \leqslant 0, \\ 0 & \text{for} \quad \bar{\theta} < -l \quad \text{and} \quad 0 < \bar{\theta}, \end{cases} \tag{4.10}$$

where the coefficient of $\bar{\theta}$ comes from the conservation of the initial area at $X = 0$ and l is regarded as a constant to be specified by the matching. The Fourier transform of $W(\bar{\theta})$ is obtained as

$$\hat{W}(\omega) = -\frac{\sqrt{2}\,\delta^2}{l^2}\frac{[1-(1+\mathrm{i}l\omega)\exp(-\mathrm{i}l\omega)]}{\omega^2} = \frac{\delta^2}{\sqrt{2}}[1+\tfrac{2}{3}l(-\mathrm{i}\omega)+O(\omega^2)]. \tag{4.11}$$

Thus we have the asymptotic expression for the tail:

$$f = \tfrac{1}{2}\delta^2\left(1+\tfrac{2}{3}l\frac{\partial}{\partial\bar{\theta}}+\dots\right)\frac{\bar{X}}{\bar{\theta}^{\frac{3}{2}}}\exp\left(-\frac{\bar{X}^2}{4\bar{\theta}}\right). \tag{4.12}$$

If only the leading term in (4.12) is taken, the asymptotic expression for the tail outside of the region is given, by using the value of $f = f_M$ at $\theta = M$, as

$$\frac{f}{f_M} = \left(\frac{M}{\theta}\right)^{\frac{3}{2}}\frac{\exp(-\psi^2)}{\exp(-\psi_M^2)} \quad \text{for} \quad \theta \geqslant M, \tag{4.13}$$

where $\psi = \delta X/(4\theta)^{\frac{1}{2}}$ and $\psi_M = \delta X/(4M)^{\frac{1}{2}}$. Taking account of this contribution to the integral of f in $\theta \geqslant M$, it is found that the conservation law is surprisingly improved. If (4.12) is taken up to the second-order term, the relative error always remains within the order of 10^{-4} at worst. In this case, the unspecified quantity l is numerically determined as

$$l = \frac{M}{1-\frac{2}{3}\psi_M^2}\left(1-\frac{f_M}{f_{M0}}\right), \tag{4.14}$$

where f_{M0} stands for the leading term in (4.12). For the initial condition (II) as well, the triangular initial condition (4.10) is reversed with respect to the origin. In this case, the asymptotic form (4.12) is unchanged and the conservation law is improved as much as in the case of (I).

For the initial conditions (III) and (IV), the asymptotic solutions must be evaluated by the saddle-point method with βX fixed. It is then found that $W(\bar{\theta})$ is now replaced by the N-wave for (III) and the triangular 'S-wave' for (IV), respectively. But the resulting lowest asymptotic expression for the tail is common to the two cases and is given by

$$\frac{f}{f_M} = \left(\frac{M}{\theta}\right)^{\frac{5}{2}}\frac{\exp(-\psi^2)}{\exp(-\psi_M^2)}. \tag{4.15}$$

It is found from this that the tail decays more quickly than (4.13). Taking account of (4.15), the corrected conservation law holds within absolute errors of order 10^{-4} in both cases.

5. Conclusion

We have examined the evolution of (1.1) asymptotically and numerically subject to the initial condition (1.2). The difference in the dissipation due to the second-order derivative and the fractional one has been highlighted. Unlike the well-known role of the former derivative, the latter fails to check the nonlinear steepening, and allows the emergence of a discontinuity. In this sense, the fractional derivative remains locally secondary in (1.1), as if it were an ordinary derivative of lower order. But because the fractional derivative is given by the hereditary integral, its effect is accumulated slowly, to spread over the whole waveform. When a shock appears, its local profile looks rounded backward compared with that in the Burgers equation. This is the very hereditary effect (after-effect) of the abrupt change which tends to continue this change. The same effect also appears in a long tail, decreasing algebraically slowly in θ, behind a localized wave. It is interesting to find that the hereditary effect makes a wavefront very smooth (flat) because it tends to maintain the undisturbed state so far experienced.

The hereditary effect can give rise to more significant dissipation than the second-order derivative in the outer region of the shock layer. This can be found from the linear dispersion relation (see (3.4)). The role of the dissipation is reversed between the high- and low-frequency limits. It is characteristic of the fractional derivative that the damping rate with respect to X is proportional to the square root of the frequency. It should also be noted that the fractional derivative can give rise to dispersion, which retards the propagation and contributes to production of a tail.

Such hereditary effects have not yet been encountered in other wave systems described by differential equations, even if some relaxation mechanisms have been taken into account. They can be clarified after solving the integro-differential equation (1.1). The solutions thus obtained can provide quantitative knowledge of the hereditary effect, especially in pursuing the far-field behaviour.

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Appendix. Non-uniformity of the expansion (2.9)

We examine the non-uniformity resulting from the divergence of V_2 in (2.9) by taking as an example the evolution from the positive step $F(\theta) = h(\theta)$. Since this condition has no characteristic scale in θ, δ can be normalized by rescaling θ and X to $\delta^2\theta\,(\equiv \bar{\theta})$ and $\delta^2 X\,(\equiv \bar{X})$, respectively. Because the relation $\mathrm{d}\tau/\mathrm{d}X = -\frac{1}{2}V$ must be satisfied at the wavefront with $V = 1$ initially, this suggests f_0 in the form

$$f_0 = h(\bar{\theta}+\tfrac{1}{2}\bar{X})+v(\bar{\theta},\bar{X}), \tag{A 1}$$

where $|v| \ll 1$. Introducing this into (2.2), the linearization yields

$$\frac{\partial v}{\partial \bar{X}}-\frac{\partial v}{\partial \bar{\theta}}=-\frac{1}{[\pi(\bar{\theta}+\frac{1}{2}\bar{X})]^{\frac{1}{2}}}-\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\bar{\theta}}\frac{1}{(\bar{\theta}-\bar{\theta}')^{\frac{1}{2}}}\frac{\partial v}{\partial \bar{\theta}'}\mathrm{d}\theta' \quad (\bar{\theta}+\tfrac{1}{2}\bar{X}>0), \tag{A 2}$$

and

$$v = 0 \quad (\bar{\theta}+\tfrac{1}{2}\bar{X}<0), \tag{A 3}$$

where we note that v may jump at $\bar{\theta} = -\frac{1}{2}\bar{X}$. Since $\bar{X} \ll 1$, we seek the asymptotic solution to (A 2):

$$v = v^{(1)}+v^{(2)}+\dots. \tag{A 4}$$

where $v^{(n+1)}/v^{(n)} \sim o(1)$ $(n = 1,2,\dots)$ as $X \to 0$. Because the leading balance in (A 2) occurs between the left-hand side and the first term on the right-hand side, we obtain

$$\left.\begin{aligned} v^{(1)} &= \frac{4}{\pi^{\frac{1}{2}}}[(\bar{\theta}+\tfrac{1}{2}\bar{X})^{\frac{1}{2}}-(\bar{\theta}+\bar{X})^{\frac{1}{2}}],\\ v^{(2)} &= 6\bar{\theta}+4\bar{X}-\frac{2}{\pi}(\sin 2\varphi-2\varphi\cos 2\varphi+4\varphi)(\bar{\theta}+\bar{X})-\frac{4\bar{X}}{\pi}\tan\varphi, \end{aligned}\right\} \tag{A 5}$$

where $\varphi = \tan^{-1}(1+2\bar{\theta}/\bar{X})^{\frac{1}{2}}$ and $-\frac{1}{2} \leqslant \bar{\theta}/\bar{X}$. At the wavefront $\bar{\theta} = -\frac{1}{2}\bar{X}$, we have $v = -(8\bar{X}/\pi)^{\frac{1}{2}}+O(\bar{X})$. Thus the decay in the strength $V(\bar{X})$ of the discontinuity at the wavefront is asymptotically given by

$$V = 1-(8\bar{X}/\pi)^{\frac{1}{2}}+O(\bar{X}). \tag{A 6}$$

Since $\mathrm{d}V/\mathrm{d}\bar{X}$ diverges as $\bar{X}\to 0$, V_2 in (2.15) would make (2.9) non-uniform if (A 6) were used for V_0. Here it is useful to note that $v^{(1)}$ is determined by the discontinuity alone, so that the leading decay in (A 6) is independent of the waveform behind the discontinuity. In other words, (A 6) is valid not only for the step function but also for other conditions given by $F(\bar{\theta}) = h(\bar{\theta})+$(regular but vanishing function at $\bar{\theta} = 0$) – for example, a triangular pulse.

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